Journal of Approximation Theory **94**, 160–171 (1998)
Article No. AT983200

Pointwise Estimate for Szasz-Type Operators*

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Communicated by Zeev Ditzian

Received May 8, 1996; accepted in revised form August 18, 1997

For combinations of modified Szasz operators D. X. Zhou gave two equivalent relations by means of the classical modulus. In this paper we extend these results by the Ditzian–Totik modulus of smoothness. © 1998 Academic Press

1. INTRODUCTION

The Szasz-type operators discussed in this paper are given by

$$L_n(f, x) = \sum_{k=0}^{\infty} n \int_0^{\infty} f(t)\, p_{n,k}(t)\, dt\, p_{n,k}(x),$$
$$p_{n,k}(x) = \frac{e^{-nx}(nx)^k}{k!}. \tag{1.1}$$

Zhou [6] considered a combination of these operators given by

$$L_{n,r}(f, x) = \sum_{i=0}^{r-1} a_i(n)\, L_{n_i}(f, x), \tag{1.2}$$

with the conditions (see [2])

$$\begin{aligned}
&\text{(a)} \quad n = n_0 < \cdots < n_{r-1} \leqslant An;\\
&\text{(b)} \quad \sum_{i=0}^{r-1} |a_i(n)| \leqslant A;\\
&\text{(c)} \quad \sum_{i=0}^{r-1} a_i(n) = 1;\\
&\text{(d)} \quad \sum_{i=0}^{r-1} a_i(n)\, n_i^{-k} = 0, \qquad \text{for} \quad k = 1, 2, ..., r-1.
\end{aligned} \tag{1.3}$$

Zhou obtained two theorems in [6].

* Supported by NSF of Hebei.

0021-9045/98 $25.00

THEOREM A. *Let* $f \in C[0, +\infty)$, $r \in N$, $0 < \alpha < r$. *Then*

$$|L_{n,r}(f, x) - f(x)| \leqslant C\left(\frac{x}{n} + n^{-2}\right)^{\alpha/2} \Leftrightarrow \omega^r(f, h) = O(h^\alpha).$$

THEOREM B. *Let* $f \in C[0, +\infty)$, $r \in N$, $0 < \alpha < r$. *We have*

$$|L_n^{(r)}(f, x)| \leqslant M\left(\min\left\{\frac{x}{n}, n^2\right\}\right)^{(r-\alpha)/2} \Leftrightarrow \omega^r(f, h) = O(h^\alpha).$$

Ditzian [1] used $\omega^2_{\varphi^\lambda}(f, t)$ and gave an interesting direct estimate for Bernstein polynomials; $\omega^r_{\varphi^\lambda}(f, t)$ was also used for polynomial approximation (see [3]). In this paper we will do this kind of work and our results contain the results of Zhou [6].

We will use some notations. Let $C[0, +\infty)$ be the set of continuous and bounded functions on $[0, +\infty)$ and

$$\omega^r_{\varphi^\lambda}(f, t) = \sup_{0 < h \leqslant t} \sup_{x \pm (rh\varphi^\lambda(\alpha)/2) \in [0, +\infty)} |\Delta^r_{h\varphi^\lambda(x)} f(x)|, \tag{1.4}$$

$$K_{\varphi^\lambda}(f, t^r) = \inf\{\|f - g\|_{C[0, +\infty)} + t^r \|\varphi^{r\lambda} g^{(r)}\|_{C[0, +\infty)}\}, \tag{1.5}$$

$$\begin{aligned}\bar{K}_{\varphi^\lambda}(f, t^r) = \inf\{&\|f - g\|_{C[0, +\infty)} + t^r \|\varphi^{r\lambda} g^{(r)}\|_{C[0, +\infty)} \\ &+ t^{r/(1-\lambda/2)} \|g^{(r)}\|_{C[0, +\infty)}\},\end{aligned} \tag{1.6}$$

where the infimum is taken on functions satisfying $g^{(r-1)} \in A \cdot C_{loc.}$, and $\varphi(x) = \sqrt{x}$, $0 \leqslant \lambda \leqslant 1$.

It is well known (see [4]) that

$$\omega^r_{\varphi^\lambda}(f, t) \sim K_{\varphi^\lambda}(f, t^r) \sim \bar{K}_{\varphi^\lambda}(f, t^r). \tag{1.7}$$

($x \sim y$ means that there exists $c > 0$ such that $c^{-1} y \leqslant x \leqslant cy$.)

Now we state our results.

If $f \in C[0, +\infty)$, $r \in N$, $0 < \alpha < r$, $0 \leqslant \lambda \leqslant 1$, then the following statements are equivalent

$$|L_{n,r}(f, x) - f(x)| = O((n^{-1/2} \delta_n^{1-\lambda}(x))^\alpha), \tag{1.8}$$

$$\omega^r_{\varphi^\lambda}(f, t) = O(t^\alpha), \tag{1.9}$$

$$\varphi^{r\lambda}(x)\, |L_n^{(r)}(f, x)| = O((n^{-1/2} \delta_n^{1-\lambda}(x))^{\alpha - r}), \tag{1.10}$$

where $\delta_n(x) = \varphi(x) + 1/\sqrt{n} \sim \max\{\varphi(x), 1/\sqrt{n}\}$.

Remark. Here we yield a generalization of Zhou's result. Naturally, as Zhou's inverse did not (and could not) cover the range between r and $2r$, the same follows here. For $\lambda = 1$, $2r$ can replace r and obtain corresponding equivalent relation of (1.8) and (1.9). It is similar to [4, (9.3.3)].

Throughout this paper C denotes a constant independent of n and x. It is not necessarily the same at each occurrence.

2. A DIRECT THEOREM

In this section we give the direct estimate of (1.9) $\Rightarrow$ (1.8).

THEOREM 1. *Let $f \in [0, +\infty)$, $r \in N$. Then we have*

$$|L_{n,r}(f, x) - f(x)| \leqslant C\omega^r_{\varphi^\lambda}(f, n^{-1/2}\delta_n^{1-\lambda}(x)). \tag{2.1}$$

Remark. If $2r$ replaces r for $\lambda = 1$, we can get a similar result of [4, (9.3.1)].

Proof. From (1.6) and (1.7) we may choose $g_n = g_{n,x,\lambda}$ for a fixed x and λ such that

$$\|f - g_n\| \leqslant C\omega^r_{\varphi^\lambda}(f, n^{-1/2}\delta_n^{1-\lambda}(x)), \tag{2.2}$$

$$n^{-r/2}\delta_n^{r(1-\lambda)}(x)\,\|\varphi^{r\lambda}g_n^{(r)}\| \leqslant C\omega^r_{\varphi^\lambda}(f, n^{-1/2}\delta_n^{1-\lambda}(x)), \tag{2.3}$$

$$(n^{-1/2}\delta_n^{1-\lambda}(x))^{r/(1-\lambda/2)}\,\|g_n^{(r)}\| \leqslant C\omega^r_{\varphi^\lambda}(f, n^{-1/2}). \tag{2.4}$$

We recall that in [6]

$$L_{n,r}((\cdot - x)^k, x) = 0, \qquad k = 1, 2, ..., r-1.$$

For u between t and x we have

$$\frac{|t-u|^{r-1}}{\varphi^{r\lambda}(u)} \leqslant \frac{|t-x|^{r-1}}{\varphi^{r\lambda}(x)}, \tag{2.5}$$

and

$$\frac{|t-u|^{r-1}}{\delta_n^{r\lambda}(u)} \leqslant \frac{|t-x|^{r-1}}{\delta_n^{r\lambda}(x)}. \tag{2.6}$$

Then by [6, (3.1)] and the Hölder inequality using (2.6), one has

$$\begin{aligned}|L_{n,r}(g_n, x) - g_n(x)| &\leqslant \left| L_{n,r}\left(\frac{1}{(r-1)!}\int_x^t (t-u)^{r-1}\, g_n^{(r)}(u)\, du, x\right)\right| \\ &\leqslant \sum_{i=1}^{r-1} |a_i(n)|\, L_{n_i}(|t-x|^r, x)\, \|\delta_n^{r\lambda} g_n^{(r)}\|\, \delta_n^{-r\lambda}(x) \\ &\leqslant Cn^{-r/2}\, \|\delta_n^{r\lambda} g_n^{(r)}\|\, \delta_n^{r(1-\lambda)}(x), \end{aligned} \tag{2.7}$$

and similarly using (2.5) we have

$$|L_{n,r}(g_n, x) - g_n(x)| \leqslant Cn^{-r/2}\delta_n^r(x)\, \varphi^{-r\lambda}(x)\, \|\varphi^{r\lambda} g_n^{(r)}\|. \tag{2.8}$$

Thus for $f \in C[0, +\infty)$, $x \in E_n = [1/n, +\infty)$, then $\delta_n(x) \sim \varphi(x)$ and by (2.2), (2.3), and (2.8) we have

$$\begin{aligned}|L_{n,r}(f, x) - f(x)| &\leqslant C(\|f - g_n\| + n^{-r/2}\delta_n^r(x)\, \varphi^{-r\lambda}(x)\, \|\varphi^{r\lambda} g_n^{(r)}\|) \\ &\leqslant C(\|f - g_n\| + n^{-r/2}\delta_n^{r(1-\lambda)}(x)\, \|\varphi^{r\lambda} g_n^{(r)}\|) \\ &\leqslant C\omega_{\varphi^\lambda}^r(f, n^{-1/2}\delta_n^{1-\lambda}(x)). \end{aligned} \tag{2.9}$$

For $x \in E_n^c = [0, 1/n)$ then $\delta_n(x) \sim 1/\sqrt{n}$, by (2.2)–(2.4) and (2.7) we have

$$\begin{aligned}&|L_{n,r}(f, x) - f(x)| \\ &\leqslant C(\|f - g_n\| + n^{-r/2}\delta_n^{r(1-\lambda)}(x)\, \|\delta_n^{r\lambda} g_n^{(r)}\|) \\ &\leqslant C[\|f - g_n\| + n^{-r/2}\delta_n^{r(1-\lambda)}(x)(\|\varphi^{r\lambda} g_n^{(r)}\| + n^{-(r\lambda)/2}\, \|g_n^{(r)}\|)] \\ &\leqslant C(\|f - g_n\| + n^{-r/2}\delta_n^{r(1-\lambda)}(x)\, \|\varphi^{r\lambda} g_n^{(r)}\| \\ &\quad + (n^{-1/2}\delta_n^{(1-\lambda)}(x))^{r/(1-\lambda/2)}\, \|g_n^{(r)}\|) \\ &\leqslant C\omega_{\varphi^\lambda}^r(f, n^{-1/2}\delta_n^{1-\lambda}(x)). \end{aligned} \tag{2.10}$$

From (2.9) and (2.10) we get (2.1).

Remark. In the case $\lambda = 0$, our result is Theorem 1 of Zhou [6].

3. AN INVERSE THEOREM

In this section we give the inverse estimate of (1.8) $\Rightarrow$ (1.9).

THEOREM 2. *Let* $f \in C[0, +\infty)$, $r \in N$, $0 < \alpha < r$, $0 \leqslant \lambda \leqslant 1$. *Then we have*

$$|L_{n,r}(f, x) - f(x)| \leqslant C(n^{-1/2}\delta_n^{1-\lambda}(x))^\alpha, \tag{3.1}$$

with a constant C independent of x and n, if and only if

$$\omega^r_{\varphi^\lambda}(f, t) = O(t^\alpha). \tag{3.2}$$

Remark. From [6] we know the term $\delta_n(x)$ cannot be replaced by $\varphi(x)$.

To prove Theorem 2 we need some new notations. Let us denote

$$C_0 := \{f \in C[0, +\infty): f(0) = 0\},$$

$$\|f\|_0 := \sup_{x \in (0, +\infty)} |\delta_n^{\alpha(\lambda-1)}(x)\, f(x)|,$$

$$C^0_\lambda := \{f \in C_0 : \|f\|_0 < \infty\},$$

$$\|f\|_r := \sup_{x \in (0, +\infty)} |\delta_n^{r+\alpha(\lambda-1)}(x)\, f^{(r)}(x)|,$$

$$C^r_\lambda := \{f \in C_0 : f^{(r-1)} \in A \cdot C_{loc.}, \|f\|_r < \infty\}.$$

We also need the following lemmas which will be proved in next section.

LEMMA 3.1. *If* $r \in N$, $0 < \alpha < r$, *then*

$$\|L_n(f)\|_r \leqslant Cn^{r/2} \|f\|_0 \qquad (f \in C^0_\lambda), \tag{3.3}$$

$$\|L_n(f)\|_r \leqslant C \|f\|_r \qquad (f \in C^r_\lambda). \tag{3.4}$$

LEMMA 3.2. *For* $0 < t < 1/8r$, $rt/2 \leqslant x \leqslant 1 - rt/2$, *and* $0 < \beta \leqslant r$, *we have*

$$\int_{-t/2}^{t/2} \cdots \int_{-t/2}^{t/2} \delta_n^{-\beta}\left(x + \sum_{j=1}^r u_j\right) du_1 \cdots du_r \leqslant Ct^r \delta_n^{-\beta}(x). \tag{3.5}$$

Proof of Theorem 2. It is sufficient to prove the inverse part. Since $L_n(f, x)$ preserves constant, hence we may assume $f \in C_0$. Suppose that (3.1) holds.

In the first place, we introduce a new K-functional as

$$K^\alpha_\lambda(f, t^r) = \inf_{g \in C^r_\lambda} \{\|f - g\|_0 + t^r \|g\|_r\}.$$

By this definition we may choose $g \in C^r_\lambda$ such that

$$\|f - g\|_0 + n^{-r/2} \|g\|_r \leqslant 2K^\alpha_\lambda(f, n^{-r/2}). \tag{3.6}$$

From (3.1) we can deduce that

$$\|L_{n,r}(f, x) - f(x)\|_0 \leqslant Cn^{-\alpha/2}.$$

Hence by Lemma 3.1 and (3.6) we have

$$
\begin{aligned}
K_\lambda^\alpha(f, t^r) &\leqslant \|f - L_{n,r}(f)\|_0 + t^r \,\|L_{n,r}(f)\|_r \\
&\leqslant Cn^{-\alpha/2} + t^r(\|L_{n,r}(f-g)\|_r + \|L_{n,r}(g)\|_r) \\
&\leqslant C[n^{-\alpha/2} + t^r(n^{r/2}\,\|f-g\|_0 + \|g\|_r)] \\
&\leqslant C\left(n^{-\alpha/2} + \frac{t^r}{n^{-r/2}} K_\lambda^\alpha(f, n^{-r/2})\right),
\end{aligned}
$$

which implies that [4, 6]

$$
K_\lambda^\alpha(f, t^r) \leqslant Ct^\alpha. \tag{3.7}
$$

On the other hand, notice that for $i = 1, ..., r$, $rt\varphi^\lambda(x)/2 \leqslant x \leqslant 1 - rt\varphi^\lambda(x)/2$. Then we get

$$
\frac{(x + (i/2)\, t\varphi^\lambda(x))^r + (x - (i/2)\, t\varphi^\lambda(x))^r}{x^r} \leqslant 2^r + 1,
$$

so for $g \in C_\lambda^0$ we have

$$
\begin{aligned}
|\Delta^r_{t\varphi^\lambda(x)}(x)| &\leqslant \|g\|_0 \left(\sum_{j=0}^r \binom{r}{j} \delta_n^{\alpha(1-\lambda)}\left(x + \left(j - \frac{r}{2}\, t\varphi^\lambda(x)\right)\right)\right) \\
&\leqslant \|g\|_0\, 2^r \left(\sum_{j=0}^r 2^{-r} \binom{r}{j} \delta_n^{2r}\left(x + \left(j - \frac{r}{2}\, t\varphi^\lambda(x)\right)\right)\right)^{\alpha(1-\lambda)/2r} \\
&\leqslant \|g\|_0\, 2^r \left(\frac{r}{2} + 1\right)(2r+1)\, \delta_n^{\alpha(1-\lambda)}(x).
\end{aligned} \tag{3.8}
$$

Using Lemma 3.2 for $g \in C_\lambda^r$, $0 < t\varphi^\lambda(x) < 1/8r$ and $rt\varphi^\lambda(x)/2 \leqslant x \leqslant 1 - rt\varphi^\lambda(x)/2$, we have

$$
\begin{aligned}
&|\Delta^r_{t\varphi^\lambda(x)}\, g(x)| \\
&\quad\leqslant \left| \int_{-(t/2)\,\varphi^\lambda(x)}^{(t/2)\,\varphi^\lambda(x)} \cdots \int_{-(t/2)\,\varphi^\lambda(x)}^{(t/2)\,\varphi^\lambda(x)} g^{(r)}\left(x + \sum_{j=1}^r u_j\right) du_1 \cdots du_r \right| \\
&\quad\leqslant \|g\|_r \int_{-(t/2)\,\varphi^\lambda(x)}^{(t/2)\,\varphi^\lambda(x)} \cdots \int_{-(t/2)\,\varphi^\lambda(x)}^{(t/2)\,\varphi^\lambda(x)} \delta^{-r+\alpha(1-\lambda)}\left(x + \sum_{j=1}^r u_j\right) du_1 \cdots du_r \\
&\quad\leqslant Ct^r\, \delta_n^{(-r+\alpha)(1-\lambda)}(x)\, \|g\|_r.
\end{aligned} \tag{3.9}
$$

From (3.7)–(3.9) for $0<t\varphi^{\lambda}(x)<1/8r$, $rt\varphi^{\lambda}(x)/2 \leqslant x \leqslant 1-rt\varphi^{\lambda}(x)/2$ and choosing appropriate g we obtain

$$\begin{aligned}|\Delta^r_{t\varphi^{\lambda}}(x)\, f(x)| &\leqslant |\Delta^r_{t\varphi^{\lambda}(x)}(f-g)(x)| + |\Delta^r_{t\varphi^{\lambda}(x)}\, g(x)| \\ &\leqslant C\delta_n^{\alpha(1-\lambda)}(x)\{\|f-g\|_0 + t^r \delta_n^{r(\lambda-1)}(x)\, \|g\|_r\} \\ &\leqslant C\delta_n^{\alpha(1-\lambda)}(x)\, K_\lambda^\alpha\left(f, \frac{t^r}{\delta_n^{r(1-\lambda)}(x)}\right) \leqslant Ct^\alpha.\end{aligned}$$

This is desirable.

Remark. If $\lambda=0$, then our result is Theorem 2 of [6].

4. THE PROOF OF THE LEMMAS

Proof of (3.3). For $x \in (0, 1/n)$, $\delta_n(x) \sim 1/\sqrt{n}$, we use the representation [5]

$$L_n^{(r)}(f, x) = n^r \sum_{k=0}^{+\infty} p_{n,k}(x) \sum_{j=0}^{r} \binom{r}{j} (-1)^j n \int_0^{+\infty} p_{n,k+r-j}(t)\, f(t)\, dt.$$

For $0 \leqslant j \leqslant r$, we consider

$$\begin{aligned}\sum_{k=0}^{+\infty} np_{n,k}(x) \int_0^{+\infty} p_{n,k+r-j}(t)\, t^r\, dt &= \left(\sum_{k=0}^{2r} + \sum_{k=2r+1}^{+\infty}\right) p_{n,k}(x) \frac{(k+2r-j)!}{n^r(k+r-j)!} \\ &:= I_1 + I_2.\end{aligned}$$

Obviously we have $I_1 \leqslant Cn^{-r}$ and

$$I_2 = \sum_{k=2r+1}^{+\infty} p_{n,k-r}(x)\, x^r \left(1+\frac{2r-j}{k}\right)\left(1+\frac{2r-j}{k-1}\right)\cdots\left(1+\frac{2r-j}{k-r+1}\right) \leqslant 3^r x^r.$$

Hence, we have

$$\begin{aligned}&|L_n^{(r)}(f, x)| \\ &\leqslant n^r \sum_{k=0}^{+\infty} p_{n,k}(x) \sum_{j=0}^{r} \binom{r}{j} n \int_0^{+\infty} p_{n,k+r-j}(t)\, \delta_n^{\alpha(1-\lambda)}(t)\, dt\, \|f\|_0 \\ &\qquad \leqslant Cn^r \sum_{j=0}^{r} \left(\sum_{k=0}^{+\infty} p_{n,k}(x)\, n \int_0^{+\infty} p_{n,k+r-j}(t) \left(t^r + \frac{1}{n^r}\right) dt\right)^{\alpha(1-\lambda)/2r} \|f\|_0 \\ &\qquad \leqslant Cn^r 2^r ((C+1)\, n^{-r} + 3^r x^r)^{\alpha(1-\lambda)/2r}\, \|f\|_0 \\ &\qquad \leqslant Cn^r \delta_n^{\alpha(1-\lambda)}(x)\, \|f\|_0.\end{aligned}$$

So we get for $x \in (0, 1/n)$

$$|\delta_n^{r+\alpha(\lambda-1)}(x)(L_n^{(r)}(f, x))| \leqslant Cn^r \delta_n^r(x) \, \|f\|_0 \leqslant Cn^{r/2} \, \|f\|_0. \tag{4.1}$$

For $x \in [1/n, +\infty)$ then $\delta_n(x) \sim \varphi(x) = \sqrt{x}$ and we use the representation (cf. [4])

$$L_n^{(r)}(f, x) = x^{-r} \sum_{i=0}^{r} Q_i(nx) \, n^i \sum_{j=0}^{+\infty} p_{n,k}(x) \left(\frac{k}{n} - x\right)^i n \int_0^{+\infty} f(t) \, p_{n,k}(t) \, dt,$$

where $Q_i(nx)$ is a polynomial in nx of degree $[(r-i)/2]$ with constant coefficients, and therefore

$$|x^{-r} Q_i(nx) \, n^i| \leqslant C \left(\frac{n}{x}\right)^{r+i/2}, \qquad \text{for} \quad x \in \left[\frac{1}{n}, +\infty\right).$$

Using the Hölder inequality we have

$$\begin{aligned}
|L_n^{(r)}(f, x)| &\leqslant C \sum_{i=0}^{r} \left(\frac{n}{\varphi^2(x)}\right)^{(r+i)/2} \sum_{k=0}^{+\infty} \left|\frac{k}{n} - x\right|^i p_{n,k}(x) \, n \\
&\quad \times \int_0^{+\infty} \delta_n^{\alpha(\lambda-1)}(t) \, p_{n,k}(t) \, dt \, \|f\|_0 \\
&\leqslant C \sum_{i=0}^{r} \left(\frac{n}{\varphi^2(x)}\right)^{(r+1)/2} \left(\sum_{k=0}^{+\infty} \left(\frac{k}{n} - x\right)^{2i} p_{n,k}(x)\right)^{1/2} \\
&\quad \times \left(\sum_{k=0}^{+\infty} p_{n,k}(x) \, n \int_0^{+\infty} \delta_n^{2r}(t) \, p_{n,k}(t) \, dt\right)^{\alpha(1-\lambda)/2r} \|f\|_0.
\end{aligned}$$

From the procedure of the proof of (4.1) we know that

$$\left(\sum_{k=0}^{+\infty} p_{n,k}(x) \, n \int_0^{+\infty} \delta_n^{2r}(t) \, p_{n,k}(t) \, dt\right)^{\alpha(1-\lambda)/2r} \leqslant C \delta_n^{\alpha(1-\lambda)}(x)$$

and recalling that [4]

$$\sum_{k=0}^{+\infty} \left(\frac{k}{n} - x\right)^{2i} p_{n,k}(x) \leqslant C n^{-i} \varphi^{2i}(x), \qquad \text{for} \quad x \in \left[\frac{1}{n}, +\infty\right).$$

Hence noticing $\delta_n(x) \sim \varphi(x)$ for $x \in [1/n, +\infty)$ we have

$$\begin{aligned}
&|\delta_n^{r+\alpha(\lambda-1)}(x) \, L_n^{(r)}(f, x)| \\
&\quad \leqslant C \delta_n^{r+\alpha(\lambda-1)}(x)(r+1) \, n^{r/2} \varphi^{-r}(x) \, \delta_n^{\alpha(1-\lambda)}(x) \, \|f_0\| \\
&\quad \leqslant C n^{r/2} \, \|f\|_0.
\end{aligned} \tag{4.2}$$

By (4.1) and (4.2) we have proved (3.3).

Proof of (3.4). By [5] we have the representation

$$L_n^{(r)}(f, x) = \sum_{k=0}^{+\infty} p_{n,k}(x)\, n \int_0^{+\infty} p_{n,k+r}(t)\, f^{(r)}(t)\, dt.$$

Hence

$$\begin{aligned}
&|L_n^{(r)}(f, x)| \\
&\leqslant \sum_{k=0}^{+\infty} p_{n,k}(x)\, n \int_0^{+\infty} p_{n,k+r}(t)\, \delta_n^{-r+\alpha(1-\lambda)}(t)\, dt\, \|f\|_r \\
&\leqslant C \left(\sum_{k=0}^{+\infty} p_{n,k}(x)\, n \int_0^{+\infty} p_{n,k+r}(t) \left(\max\left\{ t, \frac{1}{n} \right\} \right)^{-r} dt \right)^{(-r+\alpha(1-\lambda))/2r} \|f\|_r \\
&\leqslant C \left(\sum_{k=0}^{+\infty} p_{n,k}(x)\, n \int_0^{+\infty} p_{n,k+r}(t) \min\{t^{-r}, n^r\}\, dt \right)^{(-r+\alpha(1-\lambda))/2r} \|f\|_r.
\end{aligned}$$

We estimate

$$\begin{aligned}
&\sum_{k=0}^{+\infty} p_{n,k}(x)\, n \int_0^{+\infty} p_{n,k+r}(t)\, t^{-r}\, dt \\
&\qquad = \sum_{k=0}^{+\infty} p_{n,k}(x)\, n \int_0^{+\infty} n^r p_{n,k}(t) \frac{k!}{(k+r)!}\, dt \\
&\qquad = \sum_{k=0}^{+\infty} p_{n,k+r}(x)\, x^{-r} \leqslant x^{-r}.
\end{aligned}$$

So we have

$$|L_n^{(r)}(f, x)| \leqslant C(\min\{x^{-r}, n^r\})^{(-r+\alpha(1-\lambda))/2r} \|f\|_r \leqslant C\delta_n^{-r+\alpha(1-\lambda)}(x)\, \|f\|_r.$$

This is desirable.

Proof of (3.5). From [6, (4.11)], using the Hölder inequality we can deduce (3.5) easily.

5. A CONNECTION BETWEEN DERIVATIVES AND SMOOTHNESS

In this section we will give an equivalent relation between the derivatives of L_n and the modulus of smoothness which contains the results of [6] and part of the results of [5].

THEOREM 3. *Let* $f \in C[0, +\infty)$, $r \in N$, $0 \leqslant \lambda \leqslant 1$. *We have*

$$|\varphi^{\lambda r}(x)\, L_n^{(r)}(f, x)| \leqslant C n^{r/2} \delta_n^{-r(1-\lambda)}(x)\, \omega_{\varphi^\lambda}^r(f, n^{-1/2} \delta_n^{1-\lambda}(x)), \tag{5.1}$$

where $\varphi(x) = \sqrt{x}$, $\delta_n(x) = \varphi(x) + 1/\sqrt{n} \sim \max\{\varphi(x), 1/\sqrt{n}\}$.

To prove (5.1) we need the inequalities

$$|\varphi^{\lambda r}(x)\, L_n^{(r)}(f, x)| \leqslant C n^{r/2} \delta_n^{-r(1-\lambda)}(x)\, \|f\|_\infty. \tag{5.2}$$

$$|\varphi^{\lambda r}(x)\, L_n^{(r)}(f, x)| \leqslant \|\varphi^{\lambda r} f^{(r)}\|_\infty. \tag{5.3}$$

Obviously from (5.2) and (5.3) we can derive (5.1) easily.

Proof of (5.2). We discuss two cases separately.
If $x \in (0, 1/n)$, then $\delta_n(x) \sim 1/\sqrt{n}$ and by [6, (4.9)] we have

$$\begin{aligned} |\varphi^{\lambda r}(x)\, L_n^{(r)}(f, x)| &\leqslant 2^r n^r n^{(-\lambda r)/2}\, \|f\| \leqslant C n^{r/2} \left(\frac{1}{\sqrt{n}}\right)^{-r(1-\lambda)} \|f\| \\ &\leqslant C n^{r/2} \delta_n^{-r(1-\lambda)}(x)\, \|f\|. \end{aligned} \tag{5.4}$$

If $x \in (1/n, +\infty)$ then $\delta_n(x) \sim \varphi(x)$ and by [6, (4.12)] we have

$$|\varphi^{\lambda r}(x)\, L_n^{(r)}(f, x)| = \varphi^{r(\lambda-1)}(x)\, |\varphi^r(x)\, L_n^{(r)}(f, x)| \leqslant C n^{r/2} \delta_n^{-r(1-\lambda)}(x)\, \|f\|. \tag{5.5}$$

From (5.4) and (5.5) we get (5.2).
The relation (5.3) can be proved in the same way as in [6, Lemma 5.1], we omit the details. Now the proof of (5.1) is complete.

THEOREM 4. *Let* $f \in C[0, +\infty)$, $r \in N$, $0 < \alpha < 1$, $0 \leqslant \lambda \leqslant 1$. *Then*

$$|\varphi^{\lambda r}(x)\, L_n^{(r)}(f, x)| \leqslant C (n^{-1/2} \delta_n^{1-\lambda}(x))^{\alpha - r} \tag{5.6}$$

implies

$$\omega_{\varphi^\lambda}^r(f, h) = O(h^\alpha). \tag{5.7}$$

Remark. We will prove the statement as in [6]. The commutativity of the operator is also crucial in the proof here.

Proof. Let $0 < t \leqslant h < 1/8r$, $x > rt\varphi^\lambda(x)/2$. By [1] we have the commutative property.

$$L_n(L_m f)(x) = L_m(L_n f)(x), \qquad \text{for} \quad m, n \in N.$$

By Theorem 1 noting $x+(j-r/2)\,t\varphi^{\lambda}(x)\leqslant 2x$ we then have

$$
\begin{aligned}
&|\Delta^{r}_{t\varphi^{\lambda}(x)}L_m f(x)|\\
&\quad\leqslant\left|\sum_{j=0}^{r}\binom{r}{j}(-1)^{r-j}\left\{L_{n,r}\left(L_m f, x+\left(j-\frac{r}{2}\right)t\varphi^{\lambda}(x)\right)\right.\right.\\
&\qquad\left.\left.-L_m\left(f, x+\left(j-\frac{r}{2}\right)t\varphi^{\lambda}(x)\right)\right\}\right|+\left|\sum_{i=0}^{r-1}a_i(n)\,\Delta^{r}_{t\varphi^{\lambda}(x)}L_{n_i}(L_m f)(x)\right|\\
&\quad\leqslant\sum_{j=0}^{r}\binom{r}{j}c\omega^{r}_{\varphi^{\lambda}}\left(L_m f, n^{-(1/2}\delta_n^{1-\lambda}(x)\right)+\sum_{i=0}^{r-1}|a_i(n)|\\
&\qquad\times\int_{-(t/2)\varphi^{\lambda}(x)}^{(t/2)\varphi^{\lambda}(x)}\cdots\int_{-(t/2)\varphi^{\lambda}(x)}^{(t/2)\varphi^{\lambda}(x)}\left|L_m^{(r)}\left(L_{n_i}f, x+\sum_{j=1}^{r}u_j\right)\right|du_1\,du_2\cdots du_r.
\end{aligned}
$$

From (5.6) we can deduce that by $\delta_n(x)\sim\max\{\varphi(x), 1/\sqrt{n}\}$

$$|\varphi^{\lambda r}(x)\,L_n^{(r)}(f, x)|\leqslant Cn^{-(1/2)(2-\lambda)(\alpha-r)}, \tag{5.8}$$

$$|\varphi^{\lambda r}(x)\,L_n^{(r)}(f, x)|\leqslant Cn^{-(1/2)(\alpha-r)}\varphi^{(1-\lambda)(\alpha-r)}(x). \tag{5.9}$$

From (5.3), (5.8), and (5.9) we have

$$
\begin{aligned}
&\left|L_m^{(r)}\left(L_{n_i}f, x+\sum_{j=1}^{r}u_j\right)\right|\\
&\quad=\varphi^{-\lambda r}\left(x+\sum_{j=1}^{r}u_j\right)\left|\varphi^{\lambda r}\left(x+\sum_{j=1}^{r}u_j\right)L_{n_i}^{(r)}\left(f, x+\sum_{j=1}^{r}u_j\right)\right|\\
&\quad\leqslant C\varphi^{-\lambda r}\left(x+\sum_{j=1}^{r}u_j\right)n^{-(1/2)(2-\lambda)(\alpha-r)}
\end{aligned}
\tag{5.10}
$$

and

$$
\begin{aligned}
&\left|L_m^{(r)}\left(L_{n_i}f, x+\sum_{j=1}^{r}u_j\right)\right|\\
&\quad=\varphi^{-\lambda r}\left(x+\sum_{j=1}^{r}u_j\right)\left|\varphi^{\lambda r}\left(x+\sum_{j=1}^{r}x_j\right)L_{n_i}^{(r)}\left(f, x+\sum_{j=1}^{r}u_j\right)\right|\\
&\quad\leqslant C\varphi^{-r+\alpha(1-\lambda)}\left(x+\sum_{j=1}^{r}u_j\right)n^{-(1/2)(\alpha-r)}.
\end{aligned}
\tag{5.11}
$$

By [6, (4.11)] with the Hölder inequality we can easily get for $0\leqslant\beta\leqslant r$

$$\int_{-(t/2)\varphi^{\lambda}(x)}^{(t/2)\varphi^{\lambda}(x)}\cdots\int_{-(t/2)\varphi^{\lambda}(x)}^{(t/2)\varphi^{\lambda}(x)}\varphi^{-\beta}\left(x+\sum_{j=1}^{r}u_j\right)du_1\cdots du_r\leqslant C\varphi^{-\beta}(x)\,t^r\varphi^{\lambda r}(x). \tag{5.12}$$

Hence by (5.10)–(5.12) we have

$$\begin{aligned} |\Delta^r_{t\varphi^\lambda(x)} L_m f(x)| &\leqslant C\omega^r_{\varphi^\lambda}(L_m f, n^{-1/2}\delta_n^{1-\lambda}(x)) \\ &\quad + \sum_{i=0}^{r-1} |a_i(n)|\, C \min\{n^{-(1/2)(2-\lambda)(\alpha-r)} t^r, t^r n^{-(1/2)(\alpha-r)} \varphi^{(1-\lambda)(\alpha-r)}(x)\} \\ &\leqslant C\{\omega^r_{\varphi^\lambda}(L_m f, n^{-1/2}\delta_n^{1-\lambda}(x)) + t^r(n^{-1/2}\delta_n^{(1-\lambda)}(x))^{(\alpha-r)}\}. \end{aligned} \tag{5.13}$$

The following demonstration is very similar to [6]; we omit the details. From (5.13) we can obtain (5.7). The proof is complete.

ACKNOWLEDGMENTS

We are grateful to Professor Z. Ditzian and the referee for their helpful comments.

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Printed in Belgium